

# Cubic equations

A cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0 \quad \text{where } a \neq 0$$

All cubic equations have either one real root, or three real roots. In this unit we explore why this is so.

Then we look at how cubic equations can be solved by spotting factors and using a method called synthetic division. Finally we will see how graphs can help us locate solutions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- explain why cubic equations possess either one real root or three real roots
- use synthetic division to locate roots when one root is known
- find approximate solutions by drawing a graph

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# 1. Introduction

In this unit we explain what is meant by a cubic equation and how such an equation can be solved. The general strategy for solving a cubic equation is to reduce it to a quadratic equation, and then solve the quadratic by the usual means, either by factorising or using the formula.

# 2. Cubic equations and the nature of their roots

A cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0$$

It must have the term in  $x^3$  or it would not be cubic (and so  $a \neq 0$ ), but any or all of  $b$ ,  $c$  and  $d$  can be zero. For instance,

$$x^3 - 6x^2 + 11x - 6 = 0, \quad 4x^3 + 57 = 0, \quad x^3 + 9x = 0$$

are all cubic equations.

Just as a quadratic equation may have two real roots, so a cubic equation has possibly three. But unlike a quadratic equation which may have no real solution, a cubic equation always has at least one real root. We will see why this is the case later. If a cubic does have three roots, two or even all three of them may be repeated. This gives us four possibilities which are illustrated in the following examples.

### Example

Suppose we wish to solve the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

This equation can be factorised to give

$$(x - 1)(x - 2)(x - 3) = 0$$

This equation has three real roots, all different - the solutions are  $x = 1$ ,  $x = 2$  and  $x = 3$ .

In Figure 1 we show the graph of  $y = x^3 - 6x^2 + 11x - 6$ .

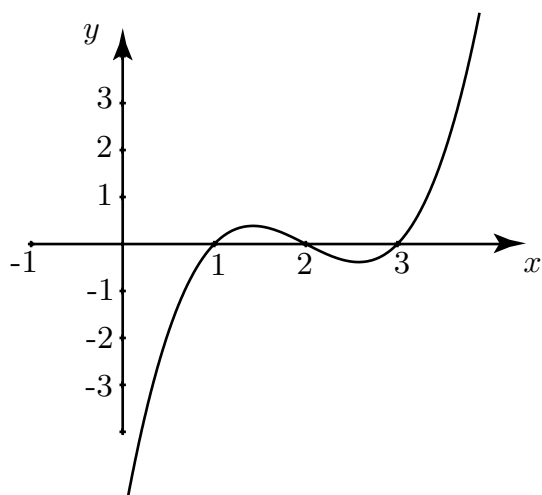


Figure 1. The graph of  $y = x^3 - 6x^2 + 11x - 6$ .

Notice that it starts low down on the left, because as  $x$  gets large and negative so does  $x^3$  and it finishes higher to the right because as  $x$  gets large and positive so does  $x^3$ . The curve crosses the  $x$ -axis three times, once where  $x = 1$ , once where  $x = 2$  and once where  $x = 3$ . This gives us our three separate solutions.

### Example

Suppose we wish to solve the equation  $x^3 - 5x^2 + 8x - 4 = 0$ .

This equation can be factorised to give

$$(x - 1)(x - 2)^2 = 0$$

In this case we do have three real roots but two of them are the same because of the term  $(x - 2)^2$ . So we only have two distinct solutions. Figure 2 shows a graph of  $y = x^3 - 5x^2 + 8x - 4$ .

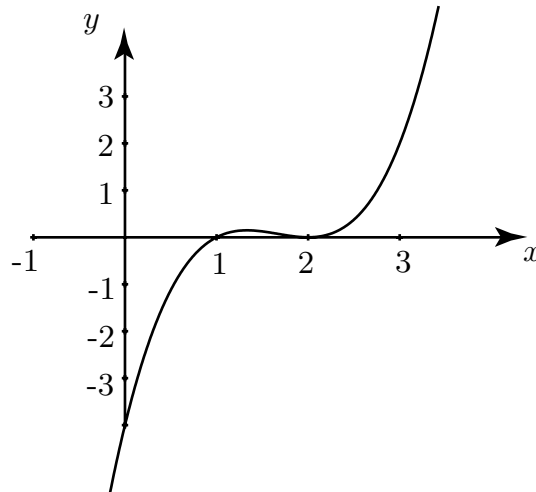


Figure 2. The graph of  $y = x^3 - 5x^2 + 8x - 4$ .

Again the curve starts low to the left and goes high to the right. It crosses the  $x$ -axis once and then just touches it at  $x = 2$ . So we have our two roots  $x = 1$  and  $x = 2$ .

### Example

Suppose we wish to solve the equation  $x^3 - 3x^2 + 3x - 1 = 0$ .

This equation can be factorised to give

$$(x - 1)^3 = 0$$

So although there are three factors, they are all the same and we only have a single solution  $x = 1$ . The corresponding curve is  $y = x^3 - 3x^2 + 3x - 1$  and is shown in Figure 3.

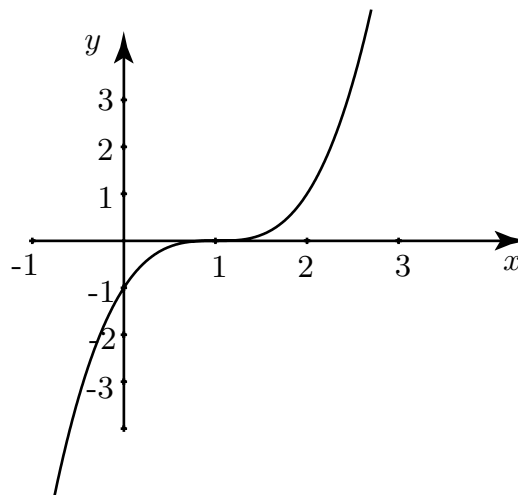


Figure 3. The graph of  $y = x^3 - 3x^2 + 3x - 1$ .

As with all the cubics we have seen so far, it starts low down on the left and goes high up to the right. Notice that the curve does cross the  $x$ -axis at the point  $x = 1$  but the  $x$ -axis is also a tangent to the curve at this point. This is indicative of the fact that there are three repeated roots.

### Example

Suppose we wish to solve the equation  $x^3 + x^2 + x - 3 = 0$ .

This equation can be factorised to give

$$(x - 1)(x^2 + 2x + 3) = 0$$

The quadratic  $x^2 + 2x + 3 = 0$  has no real solutions, so the only solution to the cubic equation is obtained by putting  $x - 1 = 0$ , giving the single real solution  $x = 1$ .

The graph  $y = x^3 + x^2 + x - 3$  is shown in Figure 4.

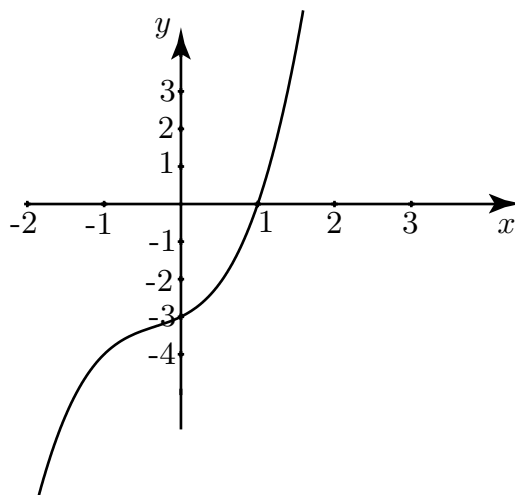


Figure 4. The graph of  $y = x^3 + x^2 + x - 3$ .

You can see that the graph crosses the  $x$ -axis in one place only.

From these graphs you can see why a cubic equation always has at least one real root. The graph either starts large and negative and finishes large and positive (when the coefficient of  $x^3$  is positive), or it will start large and positive and finish off large and negative (when the coefficient of  $x^3$  is negative).

The graph of a cubic **must** cross the  $x$ -axis at least once giving you at least one real root. So, any problem you get that involves solving a cubic equation will have a real solution.

### Exercise 1

Determine the real roots of the following cubic equations - if a root is repeated say how many times.

- a)  $(x + 1)(x - 2)(x - 3) = 0$     b)  $(x + 1)(x^2 - 12x + 20) = 0$     c)  $(x - 3)^2(x + 4) = 0$   
d)  $(x + 2)(x^2 + 6x + 10) = 0$     e)  $(x + 2)(x^2 + 7x + 10) = 0$     f)  $(x - 5)(x^2 - 10x + 25) = 0$

### 3. Solving cubic equations

Now let us move on to the solution of cubic equations. Like a quadratic, a cubic should always be re-arranged into its standard form, in this case

$$ax^3 + bx^2 + cx + d = 0$$

The equation

$$x^3 + 4x^2 - x = \frac{6}{x}$$

is a cubic, though it is not written in the standard form. We need to multiply through by  $x$ , giving us

$$x^3 + 4x^2 - x = 6$$

and then we subtract 6 from both sides, giving us

$$x^3 + 4x^2 - x - 6 = 0$$

This is now in the standard form

When solving cubics it helps if you know one root to start with.

#### Example

Suppose we wish to solve

$$x^3 - 5x^2 - 2x + 24 = 0$$

given that  $x = -2$  is a solution.

There is a theorem called the **Factor Theorem** which we do not prove here. It states that if  $x = -2$  is a solution of this equation, then  $x + 2$  is a factor of this whole expression. This means that  $x^3 - 5x^2 - 2x + 24 = 0$  can be written in the form

$$(x + 2)(x^2 + ax + b) = 0$$

where  $a$  and  $b$  are numbers.

Our task now is to find  $a$  and  $b$ , and we do this by a process called **synthetic division**. This involves looking at the coefficients of the original cubic equation, which are 1,  $-5$ ,  $-2$  and 24. These are written down in the first row of a table, the starting layout for which is

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ \hline 1 & & & & \end{array}$$

Notice that to the right of the vertical line we write down the known root  $x = -2$ . We have left a blank line which will be filled in shortly. In the first position on the bottom row we have brought down the number 1 from the first row.

The next step is to multiply the number 1, just brought down, by the known root,  $-2$ , and write the result,  $-2$ , in the blank row in the position shown.

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & & & \\ \hline 1 & & & & \end{array}$$

The numbers in the second column are then added,  $-5 + -2 = -7$ , and the result written in the bottom row as shown.

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & & & \\ \hline 1 & -7 & & & \end{array}$$

Then, the number just written down,  $-7$ , is multiplied by the known root,  $-2$ , and we write the result,  $14$ , in the blank row in the position shown.

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & 14 & & \\ \hline 1 & -7 & & & \end{array}$$

Then the numbers in this column are added:

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & 14 & & \\ \hline 1 & -7 & 12 & & \end{array}$$

The process continues:

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & 14 & -24 & \\ \hline 1 & -7 & 12 & 0 & \end{array}$$

Note that the final number in the bottom row (obtained by adding  $24$  and  $-24$ ) is zero. This is confirmation that  $x = -2$  is a root of the original cubic. If this value turns out to be non-zero then we do not have a root.

At this stage the coefficients in the quadratic that we are looking for are the first three numbers in the bottom row. So the quadratic is

$$x^2 - 7x + 12$$

So we have reduced our cubic to

$$(x + 2)(x^2 - 7x + 12) = 0$$

The quadratic term can be factorised to give

$$(x + 2)(x - 3)(x - 4) = 0$$

giving us the solutions  $x = -2, 3$  or  $4$

In the previous Example we were given one of the roots. If a root is not known it's always worth trying a few simple values.

### Example

Suppose we wish to solve

$$x^3 - 7x - 6 = 0$$

A very simple value we might try is  $x = 1$ . Substituting  $x = 1$  in the left-hand side we find

$$1^3 - 7(1) - 6$$

which equals  $-12$ , so this value is clearly not a solution. Suppose we try the value  $x = -1$ :

$$(-1)^3 - 7(-1) - 6$$

which does equal zero, so  $x = -1$  is a solution. This means that  $x + 1$  is a factor and the cubic can be written in the form

$$(x + 1)(x^2 + ax + b) = 0$$

We can perform synthetic division to find the other factors.

As before, we take the coefficients of the original cubic equation, which are 1, 0,  $-7$  and  $-6$ . These are written down in the first row of a table, the starting layout for which is

$$\begin{array}{cccc|c} 1 & 0 & -7 & -6 & x = -1 \\ \hline 1 & & & & \end{array}$$

To the right of the vertical line we write down the known root  $x = -1$ . In the first position on the bottom row we have brought down the number 1 from the first row.

The next step is to multiply the number 1, just brought down, by the known root,  $-1$ , and write the result,  $-1$ , in the blank row in the position shown.

$$\begin{array}{cccc|c} 1 & 0 & -7 & -6 & x = -1 \\ & -1 & & & \\ \hline 1 & & & & \end{array}$$

The numbers in the second column are then added,  $0 + -1 = -1$ , and the result written in the bottom row as shown.

$$\begin{array}{cccc|c} 1 & 0 & -7 & -6 & x = -1 \\ & -1 & & & \\ \hline 1 & -1 & & & \end{array}$$

Then, the number just written down,  $-1$ , is multiplied by the known root,  $-1$ , and we write the result, 1, in the blank row in the position shown.

$$\begin{array}{cccc|c} 1 & 0 & -7 & -6 & x = -1 \\ & -1 & 1 & & \\ \hline 1 & -1 & & & \end{array}$$

Then the numbers in this column are added:

$$\begin{array}{cccc|c} 1 & 0 & -7 & -6 & x = -1 \\ & -1 & 1 & & \\ \hline 1 & -1 & -6 & & \end{array}$$

The process continues:

$$\begin{array}{cccc|c} 1 & 0 & -7 & -6 & x = -1 \\ & -1 & 1 & 6 & \\ \hline 1 & -1 & -6 & 0 & \end{array}$$

At this stage the coefficients in the quadratic that we are looking for are the first three numbers in the bottom row. So the quadratic is

$$x^2 - x - 6$$

So we now need to solve the equation

$$(x + 1)(x^2 - x - 6) = 0$$

which factorises to give

$$(x + 1)(x - 3)(x + 2) = 0$$

and the three solutions to the cubic equation are  $x = -2, -1$  or  $3$ .

Sometimes you may be able to spot a factor as shown in the following Example.

### Example

Suppose we wish to solve the equation  $x^3 - 4x^2 - 9x + 36 = 0$ .

Observe that the first two terms in the cubic can be factorised as  $x^3 - 4x^2 = x^2(x - 4)$ . The second pair of terms can be factorised as  $-9x + 36 = -9(x - 4)$ . This sort of observation can only be made when you have had sufficient practice and experience of handling expressions like this. However, the observation enables us to proceed as follows:

$$\begin{aligned} x^3 - 4x^2 - 9x + 36 &= 0 \\ x^2(x - 4) - 9(x - 4) &= 0 \end{aligned}$$

The common factor of  $(x - 4)$  can be extracted to give

$$(x^2 - 9)(x - 4) = 0$$

and the difference of two squares can be factorised giving

$$(x + 3)(x - 3)(x - 4) = 0 \tag{1}$$

giving us solutions  $x = -3, 3$  or  $4$

You may have noticed that in each example we have done, every root is a factor of the constant term in the equation. In the last Example,  $-3, 3$  and  $4$  all divide into the constant term  $36$ . As long as the coefficient of  $x^3$  in the cubic equation is  $1$  this must be the case. This is because, referring to equation (1) for example, the constant term arises from multiplying the numbers  $3, -3$  and  $-4$ . This gives us another possible approach.



### Example

Suppose we wish to solve the equation

$$x^3 - 6x^2 - 6x - 7 = 0$$

If there is to be a solution, then because the coefficient of  $x^3$  is 1, it is going to be an integer and it's going to be a factor of 7. This leaves us with only four possibilities: 1, -1, 7, and -7. So we can try each of them in turn. You can see fairly quickly that 1 and -1 don't work. So we will try 7. Rather than substitute  $x = 7$  into the cubic equation we shall immediately try to synthetically divide this expression by 7.

$$\begin{array}{cccc|c} 1 & -6 & -6 & -7 & x = 7 \\ & 7 & 7 & 7 & \\ \hline 1 & 1 & 1 & 0 & \end{array}$$

Because the fourth number in the last row is zero it follows that  $x = 7$  is indeed a root. Had this number not been zero then  $x = 7$  would not have been a root. So 7 is indeed a root, and the resulting quadratic is

$$x^2 + x + 1 = 0$$

Now you will find that if you try to solve it that the quadratic equation  $x^2 + x + 1 = 0$  has no real solutions, so the only possible solution to this cubic is  $x = 7$ .

Certain basic identities which you may wish to learn can help in factorising both cubic and quadratic equations.

### Example

Suppose we wanted to solve the equation  $x^3 + 3x^2 + 3x + 1 = 0$ .

This has the widely-known factorisation  $(x + 1)^3 = 0$  from which we have the root  $x = -1$  repeated three times.

### Exercise 2

1. For each of the following cubic equations one root is given. Determine the other roots of each cubic.

a)  $x^3 + 3x^2 - 6x - 8 = 0$  has a root at  $x = 2$ .

b)  $x^3 + 2x^2 - 21x + 18 = 0$  has a root at  $x = 3$ .

c)  $x^3 + 4x^2 + 7x + 6 = 0$  has a root at  $x = -2$ .

d)  $2x^3 + 9x^2 + 3x - 4 = 0$  has a root at  $x = -4$ .

2. For each of the following cubic equations use synthetic division to determine if the given value of  $x$  is a root of the equation. Where it is, determine the other roots of the equation.

a) Is  $x = -2$  a root of the equation  $x^3 + 9x^2 + 26x + 24 = 0$ ?

b) Is  $x = 4$  a root of the equation  $x^3 - 6x^2 + 9x + 1 = 0$ ?

c) Is  $x = -1$  a root of the equation  $x^3 + 6x^2 + 3x - 5 = 0$ ?

d) Is  $x = 2$  a root of the equation  $x^3 + 2x^2 - 20x + 24 = 0$ ?

## 4. Using graphs to solve cubic equations

If you cannot find a solution by these methods then draw an accurate graph of the cubic expression. The points where it crosses the  $x$ -axis will give you the solutions to the equation but their accuracy will be limited to the accuracy of your graph.

You may indeed find that the graph crosses the  $x$ -axis at a point that would suggest a factor, for instance, if you draw a graph that appears to cross the  $x$ -axis at, say,  $x = \frac{1}{2}$ , then it's worth trying to find out whether  $\left(x - \frac{1}{2}\right)$  is indeed a factor.

### Example

Suppose we wished to solve  $x^3 + 4x^2 + x - 5 = 0$ .

Now this equation will not yield a factor by any of the methods that we have discussed. So a graph of  $y = x^3 + 4x^2 + x - 5$  has been drawn as shown in Figure 5.

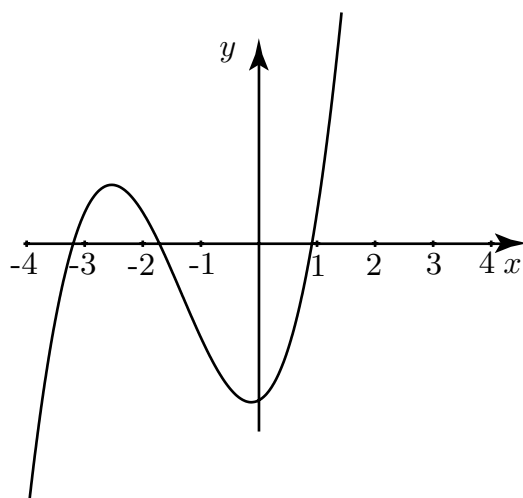


Figure 5. A graph of  $y = x^3 + 4x^2 + x - 5$

It crosses the  $x$ -axis at three places and hence there are three real roots. As we stated before their accuracy will be limited to the accuracy of the graph. From the graph we find the approximate solutions  $x \approx -3.2, -1.7, 0.9$ .

### Exercise 3

Use graphs to solve, as accurately as you can, the following cubic equations.

a)  $x^3 - 4x^2 - 6x + 5 = 0$

b)  $2x^3 - 3x^2 - 4x - 35 = 0$

c)  $x^3 - 3x^2 - x + 1 = 0$

## Answers

### Exercise 1

- a)  $-1, 2, 3$    b)  $-1, 2, 10$    c)  $-4, 3$  (twice)  
d)  $-2$    e)  $-5, -2$  (twice)   f)  $5$  (three times)

### Exercise 2

- a)  $x = -1$  and  $x = -4$    b)  $x = -6$  and  $x = 1$   
c) No other roots   d)  $x = -1$  and  $x = 0.5$
- a)  $x = -2$  is a root. Other roots are  $x = -3$  and  $x = -4$   
b)  $x = 4$  is not a root  
c)  $x = -1$  is not a root  
d)  $x = 2$  is a root. Other roots are  $x = -6$  and  $x = 2$

### Exercise 3

All answers are correct to 2 significant figures.

- a)  $-1.6, 0.62, 5$    b)  $3.5$ , no other real roots   c)  $-0.68, 0.46, 3.2$